

Preferential attachment with information filtering - node degree probability distribution properties

Hrvoje Štefančić^a Vinko Zlatić^a

^a*Theoretical Physics Division, Rudjer Bošković Institute*

P.O.B. 180, HR-10002 Zagreb, Croatia

Abstract

A network growth mechanism based on a two-step preferential rule is investigated as a model of network growth in which no global knowledge of the network is required. In the first filtering step a subset of fixed size m of existing nodes is randomly chosen. In the second step the preferential rule of attachment is applied to the chosen subset. The characteristics of thus formed networks are explored using two approaches: computer simulations of network growth and a theoretical description based on a master equation. The results of the two approaches are in excellent agreement. Special emphasis is put on the investigation of the node degree probability distribution. It is found that the tail of the distribution has the exponential form given by $\exp(-k/m)$. Implications of the node degree distribution with such tail characteristics are briefly discussed.

Key words: Scale-free networks, Master equation, Degree distribution

PACS: 89.75.Hc, 89.75.Da, 05.10.Gg, 02.50.Ng

1 Introduction

In many aspects of the society and the world we live in, network structures are a common occurrence. At the level of their phenomenological description, these networks are very heterogeneous both in the nature of their nodes and their links and, apart from their very network nature, seemingly have nothing in common. However, more detailed statistical analysis of various communication, transport, social, biological and other networks have revealed amazing similarities in their statistical characteristics. These findings, implying the existence of a general underlying organizing principle, have stimulated an entirely new wave of research and have given birth to the field of complex networks [1,2,3,4].

The statistical characteristic of complex networks that proved to be the most suitable for demonstrating the common features of very different networks is the distribution of node degrees. It was found that in complex networks as different as Internet [5], world wide web [6], citations [7], scientific collaborations [8], metabolic interactions [9] and many others, the distribution of degrees has a power law character for large values of node degrees, i.e. a power law tail. This property of the aforementioned networks, which are also referred to as *scale-free* networks, was shown to be important in dynamical processes on networks, such as human epidemics and e-mail virus spreading [10,11] and resilience to Internet attacks [12,13]. A deeper understanding and modeling of laws behind the formation of scale-free networks was clearly necessary.

Email addresses: `shrvoje@thphys.irb.hr` (Hrvoje Štefančić), `vzlatic@irb.hr` (Vinko Zlatić).

The first crucial step towards unraveling the principles behind “scale-free” networks was undertaken by Barabási and Albert [14]. Their model (in further text referred to as BA model) is founded on two simple, but essential principles [15]: network growth and preferential attachment. The network growth is realized by adding a new node to the network at each time step. The new node attaches to r already existing nodes in the network. The rule of preferential attachment dictates that the probability of attaching a new node to the existing node is proportional to the degree of the existing node. These two basic principles are sufficient to reproduce a power law tail. A host of modifications and elaborations of the preferential attachment rule in the evolving networks followed very soon after the formulation of the BA model [16,17,18,19]. These models succeeded in reproducing some of the statistical characteristics of complex networks, such as the power law exponent, in a more realistic manner.

Apart from the preferential attachment rule, it is possible to question how other elements of the BA model could be modified to better capture the properties of real world networks. In the process of adding a new node to a very large network (such as www), it is reasonable to expect that only part of information about the entire network is effectively available. Such an assumption is grounded in the fact that nodes usually have a limited processing capability or have (e.g. topical) interest only in a part of the network. It is also reasonable to expect that within the processed part of the network the attachment rule of the new node to the old ones is not completely random, but is preferential. Namely, some of the old nodes are more “appealing” to the new ones. A reasonable variant which encompasses the aforementioned properties is the filtration of global information [20]. In this model of network growth, a new node selects a subset of existing nodes and then applies the rule of preferential

attachment to the selected subset. The principle of selection may be chosen in various ways. The subset may be formed by randomly choosing a finite fraction or a fixed number of existing nodes [20]. In the case of the choice of the finite fraction, however, the selected subset may become unrealistically large for large networks and moderate fractions. One can also introduce an additional attribute of every node, and select into the subset only those existing nodes the attributes of which are sufficiently similar to the attribute of the new node [21]. This model may, e.g. simulate the attachment to topically similar pages on the www.

The investigation of the model of preferential attachment based on the realistic access to the information on the network structure is also important with respect to the properties of dynamical processes on networks. It has been shown [10] that in scale-free networks there exists no threshold for the spreading of epidemics in human society or computer viruses in the Internet. The key feature for the nonexistence of the epidemics spreading threshold (or equivalently, lack of the phase transition) is the power law tail of the node degree distribution in scale-free networks. In the realistic modifications of information filtering it is reasonable to expect that some of the characteristics of the node degree distribution might change. One of the aims of this paper is to determine the behavior of the node degree distribution function for large degrees and its consequences for the dynamical processes on networks formed by information filtering.

In this paper, we study a particular model of information filtering with special emphasis on its theoretical description and tail characteristics. In this model, already introduced in [20], a selected subset, which we call the filtration subset, is formed by randomly choosing a fixed number of existing nodes. We concen-

trate on the determination of the node degree distribution and approach the problem in two different ways: theoretically, applying the master equation approach of Dorogovtsev and Mendes [23] and using computer simulations, in a manner analogous to [20]. The results of these two approaches are compared and their agreement discussed in detail.

2 Model

We start with the discussion of computer simulations of the network growth in our model. The network grows by addition of one new node at each time step. Each new node connects to the existing network with *one* link. This special choice facilitates the theoretical treatment and ensures a higher level of analytical tractability. Therefore, at the time step t , there are t links in the network. In every simulation of the network growth, a core of the network is initially formed. The core of the network is formed in a growth procedure where each new node is randomly connected to one of the existing nodes. The size of the network core is chosen to be larger than the maximal size of the filtration subset that is considered in this paper. As we consider filtration subset sizes up to 1000, the network core size is taken to be 1100 in all simulations. The nodes are numbered starting from 0, and the network is allowed to grow until the number of nodes, and respectively the number of links, reaches the maximal value n_{max} , which is generally taken as $n_{max} = 10^6$. The statistical descriptors of the investigated network are obtained as averages over 100 independent realizations of the network topology. The stability of the node degree probability distribution was tested by varying the final network size from 10^5 to 10^7 . The simulations with different values of n_{max} show the stability of the

node degree probability distribution for all but the largest node degrees. The probability distributions are characterized by the exponential cut-off in the tail. For larger n_{max} , the position of the cut-off is shifted more towards larger node degrees, as already found in [20]. The stability issues will further be addressed in section 4. It has also been found that the node degree probability distribution does not depend in any significant way on the choice of the core of the simulated network. The results of simulations are displayed in Fig. A.1. The cumulative probability distribution clearly exhibits the power law scaling for a broad range of node degrees and has a rapidly falling tail. The exponent of the power law behavior of the cumulative probability distribution is close to -2.

3 Theoretical treatment

In the theoretical treatment of the node degree distributions we adopt the master equation approach of [23]

In the type of the growing network investigated in this paper, each node is uniquely identified by the time of its attachment to the network which is hereafter denoted by s . In such a framework, we introduce the probability that a node created at time s , at some time t , where $t \geq s$, has the degree k and denote it as $p(k, s, t)$. The probability that a randomly chosen node has a degree k at time t is denoted by $P(k, t)$. The cumulative probability distribution is denoted by $P_{cum}(k, t)$ and defined as $P_{cum}(k, t) = \sum_{l=k}^{\infty} P(l, t)$. The quantity of interest in our theoretical approach is the probability that the node added at time t attaches to the node created at time s which at that time has the degree k . The size of the filtration subset is m . The probability

that the chosen filtration subset at time t contains the node created at s with the degree k and that the sum of degrees of all nodes in the subset equals l is

$$w(l-k, m, t) = \frac{\binom{t-1}{m-1} \cdot g(l-k, t)}{\binom{t}{m}}, \quad (1)$$

where $g(u, t)$ is the probability that the sum of degrees of $m-1$ nodes is u at time t . It is obtained as a convolution of the order $m-1$ of the probability function $P(k, t-1)$, i.e.

$$g(u, t) = P^{*(m-1)}(u, t-1). \quad (2)$$

The expression (2) allows us to write equation (1) as

$$w(l-k, m, t) = \frac{m}{t} P^{*(m-1)}(l-k, t-1). \quad (3)$$

This probability of choosing the desired node in a selected filtration subset must be multiplied by the probability of choosing the node created at time s from the selected filtration set. This phase of selection is preferential, i.e. the probability for any node being chosen is proportional to its degree. For the selected subset described above, this probability amounts to k/l . With all necessary elements specified, we can formulate the master equation for the probabilities $p(k, s, t)$:

$$p(k, s, t) = \left(\sum_{l=k-1+m-1}^{k-1+a_{max}} w(l-(k-1), m, t) \frac{k-1}{l} \right)$$

$$\begin{aligned}
& \times p(k-1, s, t-1) \\
& + \left(1 - \sum_{l=k+m-1}^{k+a_{max}} w(l-k, m, t) \frac{k}{l} \right) \\
& \times p(k, s, t-1),
\end{aligned} \tag{4}$$

for $k \geq 2$ and

$$\begin{aligned}
p(1, s, t) &= \delta_{s,t} + (1 - \delta_{s,t}) \left(1 - \sum_{l=m}^{1+a_{max}} w(l-1, m, t) \frac{1}{l} \right) \\
& \times p(1, s, t-1).
\end{aligned} \tag{5}$$

The limits of summation take into account the fact that the minimal degree of any node in the network is 1. The term in the upper limit of the summations, denoted by a_{max} , specifies the maximal value the sum of $m-1$ degrees can have. In the limit of infinitely large networks, it becomes infinite. Performing index substitutions in the summations on the right-hand side of (4) and (5) and using (3), we obtain

$$\begin{aligned}
p(k, s, t) &= \frac{k-1}{t} \left(m \sum_{r=m-1}^{a_{max}} \frac{P^{*(m-1)}(r, t-1)}{r+k-1} \right) \\
& \times p(k-1, s, t-1) \\
& + \left(1 - \frac{k}{t} \left(m \sum_{r=m-1}^{a_{max}} \frac{P^{*(m-1)}(r, t-1)}{r+k} \right) \right) \\
& \times p(k, s, t-1)
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
p(1, s, t) &= \delta_{s,t} + \left(1 - \frac{m}{t} \sum_{r=m-1}^{a_{max}} \frac{P^{*(m-1)}(r, t-1)}{r+1} \right) \\
& \times p(1, s, t-1).
\end{aligned} \tag{7}$$

The probability function for the node degree can be further defined in terms of the probability $p(k, s, t)$ as

$$P(k, t) = \frac{1}{t+1} \sum_{s=0}^t p(k, s, t). \quad (8)$$

Furthermore, we introduce a function

$$f_m(k, t) = m \sum_{r=m-1}^{a_{max}} \frac{P^{*(m-1)}(r, t-1)}{r+k}, \quad (9)$$

which, together with (8), leads to the recursive relations for the node degree probability function:

$$\begin{aligned} P(k, t) &= \frac{k-1}{t-1} f_m(k-1, t) P(k-1, t-1) \\ &+ \left(1 - \frac{k}{t} f_m(k, t)\right) \frac{t}{t+1} P(k, t-1) \end{aligned} \quad (10)$$

and

$$P(1, t) = \frac{1}{t+1} + \left(1 - \frac{1}{t} f_m(1, t)\right) \frac{t}{t+1} P(1, t-1). \quad (11)$$

The set of equations displayed so far operates with the time-dependent probability distributions. As the results of simulations imply, the node degree probability distribution becomes stable when the growing network becomes large enough. Therefore, in further considerations we investigate the stable probability distribution. This implies that $P(k, t) \rightarrow P(k)$ and $f_m(k, t) \rightarrow f_m(k)$. For the stable degree node probability distribution, we have the following relations:

$$P(k) = \frac{(k-1) f_m(k-1) P(k-1)}{1 + k f_m(k)} \quad (12)$$

and

$$P(1) = \frac{1}{1 + f_m(1)} . \quad (13)$$

The set of equations (12) and (13) is a complicated set of equations given the fact that the functions $f_m(k)$ depend on the entire probability distribution $P(k)$ in a nonlinear fashion. In solving this set of equations we adopt the following iterative procedure. The function $f_m(k)$ is calculated using a guess initial node degree probability distribution of the form $P(k) \sim k^{-\gamma}$ with $\gamma = 3$ (reminiscent of the Albert-Barabási model). The function $f_m(k)$ thus obtained, is further used to recursively calculate the probability distribution $P(k)$ from the set of equations (12) and (13). The obtained probability distribution is then used to calculate the function $f_m(k)$ and the entire procedure is repeated until satisfactory convergence is achieved.

4 Results

The results of the calculations of the theoretical node degree probability distribution show excellent agreement with the analogous results obtained in simulations. In Figs A.2, A.3, A.4, and A.5 we show graphs of cumulative degree node probability distributions for the values of the subset size $m = 2, 10, 100$, and 1000 . The agreement of the two cumulative distributions obtained by completely different procedures is excellent, except for the largest values of the node degrees. The disagreement in this area of k is attributed to the finite size effects of the cumulative distribution obtained by simulation. The agreement between the theoretical and simulational cumulative probability distribution is present not only for the linear parts of the graphs in the log-log plots, but

also for the curved ones which represent the declination from the power law i.e. the quickly decaying tail. This fact is especially clear for smaller values of the filtering subset m (see e.g. Figs A.2 and A.3) Therefore, the theoretical distribution functions are capable of explaining the characteristics of the tail of the cumulative probability distribution $P_{cum}(k)$. These findings support the use of the theoretically constructed distributions as predictions of the results that would be obtained in the simulations of larger networks. The stability of the node degree probability distribution for $m = 10$ is shown in Fig. A.6. For larger sizes of the simulated network n_{max} , the node degree probability distribution approaches more to the theoretically obtained one. From the distributions displayed in Fig. A.6 it is natural to expect that the simulated and theoretical curves will be in full agreement for the infinitely large simulated network. An analogous behavior is observed for all sample sizes m . The disagreements in the tails of probability distributions between finite size simulated networks and theory are attributed to the finite size of the simulated networks.

Inspection of the theoretically constructed cumulative probability distribution shows that this distribution has a tail decreasing faster than the power law which dominates the appearance of the distribution for quite a large number of decades in k . Further inspection of these graphs in a linear-log plots reveals the exponential nature of the distribution tails. Let us further examine characteristics of these tails. Interesting information in this direction can be provided by taking into consideration the function $f_m(k)$ appearing in the recursion relation for the probability function $P(k)$. In Fig. A.7 it is clearly

shown that $1/f_m$ is very linear, i.e. the form of this function is

$$f_m(k) = \frac{1}{a_m + b_m k}. \quad (14)$$

Based on this form of dependence of f_m on k , we can treat the set of equations (12) and (13) in a much more analytically tractable fashion. For large values of k it is possible to write Eq. (12) in a continuum approximation as a differential equation

$$P(k) = -\frac{d}{dk} \left(\frac{k}{a_m + b_m k} P(k) \right), \quad (15)$$

or, equivalently as

$$k \frac{dP(k)}{dk} = \left(\frac{b_m k}{a_m + b_m k} - a_m - b_m k - 1 \right) P(k). \quad (16)$$

The solution of Eq. (16) is of the form

$$P(k) = C(a_m + b_m k) k^{-(a_m+1)} e^{-b_m k}, \quad (17)$$

where C is the integration constant. The cumulative probability distribution, obtained by the integration of expression (17), then acquires the form

$$P_{cum}(k) \sim k^{-a_m} e^{-b_m k}. \quad (18)$$

The final result demonstrates that the tail of the distribution has the exponential nature which is governed by the value of the coefficient b_m . The value of the coefficient a_m determines the exponent of the power law part of the cumulative probability function form. Figure A.8, displaying the dependence of the coefficients a_m and b_m on m , shows that for $m \geq 5$, the value of the coefficient a_m lies in the vicinity of 2, which produces the power law form analogous to

the original Albert-Barabási power law. The larger the value of m , the closer is a_m to the value 2. The coefficient b_m falls with the increase of the subset size m . Its dependence on m is given by the regression $b_m = 1.0003 m^{-0.99982}$.

We see that a simple formula

$$b_m = \frac{1}{m} \quad (19)$$

describes the dependence of b_m on m very well. Let us consider this fact in more detail to gain further understanding of the characteristics of the function f_m and its coefficients a_m and b_m . The convolution function $P^{*(m-1)}(k)$ is a rather well localized distribution function, irrespectively of the (non)localized character of the probability distribution function $P(k)$. This kind of behavior of the convolution probability function is illustrated in Fig. A.9. The function $f_m(k)$ can then be written in the form

$$f_m(k) = \sum_{r=m-1}^{a_{max}} \frac{P^{*(m-1)}(r)}{\frac{r}{m} + \frac{k}{m}}. \quad (20)$$

The convolution probability distribution $P^{*(m-1)}(k)$ is localized around its average value, amounting to $(m-1)\bar{k}$, where \bar{k} is the average node degree in the network. The largest contribution to the function $f_m(k)$ comes from the interval around the average value of the convolution probability distribution $P^{*(m-1)}(k)$. Assuming that the denominator in the sum of the expression does not vary much in the interval where the convolution probability distribution is non-negligible, we can approximate r in the denominator of this expression with $(m-1)\bar{k}$. Now it is possible to trivially sum the terms in the numerator to 1 (since it is a probability function) and the expression (20) becomes

$$f_m(k) = \frac{1}{\frac{(m-1)\bar{k}}{m} + \frac{k}{m}}. \quad (21)$$

This chain of arguments validates the expression (14) and explains its origin. It is interesting to note, as specified in Fig. A.7, that the assumptions introduced in the preceding paragraph work satisfactorily well even for the smallest values of the parameter m .

5 Conclusion

In conclusion, we have investigated the modification of the BA model with the fixed size filtration subset. The node degree probability distribution was studied using simulations of network growth and theoretical modeling. These two approaches yield results for the node degree probability distribution which are in excellent agreement. The theoretical approach enables one to gain a deeper insight into the characteristics of the distribution. The cumulative probability distribution has a power law character with an exponential tail. The power law exponent approaches the BA model value -2 as the filtration subset size m becomes larger. Furthermore, excellent agreement of the simulational and theoretical distributions for all node degrees except the largest, enables us to use theoretical distribution in gaining information on the distribution tail beyond information available from simulations which are constrained by the finite sizes of the simulational data sets. The tail decays exponentially, the decay being slower for the larger values of m . More precisely, the exponential factor is of the form $e^{-b_m k}$, where $b_m = 1/m$ is a very accurate description. The results reported in this paper demonstrate the possible mechanism of the realistic access to the network information in the process of growth of the network by preferential attachment. It is clearly shown how the declination from the concept of the global knowledge of the network leads to the modifications

in the node degree probability distribution and how the exponential tail appears. These results have interesting implications on the problem of epidemics spreading since the exponential tail in the node degree probability function results in the existence of non-vanishing epidemics threshold. All other dynamic processes on networks that are sensitive to the (in)finiteness of the moments of the node degree probability distribution are affected as well.

The modification of the preferential attachment rule considered in this paper is just one of the possible elaborations towards the understanding of realistic networks. Given the omnipresence of network structures and their large impact on our lives and societies, further investigations of their formation mechanism are called for.

Acknowledgments. The authors would like to thank A. Šiber for useful comments on the manuscript. This work was supported by the Ministry of Science and Technology of the Republic of Croatia under the contract numbers 0098002 and 0098004.

References

- [1] R. Albert, and Barabási, A.-L., *Rev. Mod. Phys.* **74**, 47–97 (2002).
- [2] Dorogovtsev, S. N. and Mendes, J. F. F., *Advances in Physics* **51**, 1079–1187 (2002).
- [3] M.E.J. Newman, *SIAM Review* **45**, 167 (2003).
- [4] Watts, D. J. and Strogatz, S. H., *Nature* **393**, 440–442 (1998).
- [5] Chen, Q., Chang, H., Govindan, R., Jamin, S., Shenker, S. J., and Willinger,

- W., in *Proceedings of the 21st Annual Joint Conference of the IEEE Computer and Communications Societies*, IEEE Computer Society (2002).
- [6] Broder, A., Kumar, R., Maghoul, F., Raghavan, P., Rajagopalan, S., Stata, R., Tomkins, A., and Wiener, J., *Computer Networks* **33**, 309–320 (2000).
 - [7] Redner, S., *Eur. Phys. J. B* **4**, 131–134 (1998).
 - [8] Grossman, J. W. and Ion, P. D. F., *Congressus Numerantium* **108**, 129–131 (1995).
 - [9] Jeong, H., Mason, S., Barabási, A.-L., and Oltvai, Z. N., *Nature* **411**, 41–42 (2001).
 - [10] Pastor-Satorras, R. and Vespignani, A., *Phys. Rev. Lett.* **86**, 3200–3203 (2001).
 - [11] Pastor-Satorras, R. and Vespignani, A., *Phys. Rev. E* **63**, 066117 (2001).
 - [12] Albert, R., Jeong, H., and Barabási, A.-L., *Nature* **406**, 378–382 (2000).
 - [13] Cruciti, P., Latora, V., Marchiari, M., Rapisarda, A., *Physica A* **320**, 642 (2003).
 - [14] Barabási, A.-L. and Albert, R., *Science* **286**, 509–512 (1999).
 - [15] Barabási, A.-L., Albert, R., and Jeong, H., *Physica A* **272**, 173–187 (1999).
 - [16] Bianconi, G. and Barabási, A.-L., *Phys. Rev. Lett.* **86**, 5632–5635 (2001).
 - [17] Bianconi, G. and Barabási, A.-L., *Europhys. Lett.* **54**, 436–442 (2001).
 - [18] Krapivsky, P. L. and Redner, S., *Phys. Rev. E* **63**, 066123 (2001).
 - [19] Krapivsky, P. L., Redner, S., and Leyvraz, F., *Phys. Rev. Lett.* **85**, 4629–4632 (2000).
 - [20] S. Mossa, M. Barthélémy, H.E. Stanley and L.A.N. Amaral, *Phys. Rev. Lett.* **88**, 138701 (2002).
 - [21] J. Gomez-Gardenes, Y. Moreno, cond-mat/0401269.

[22] Dorogovtsev, S. N. and Mendes, J. F. F., *Europhys. Lett.* **50**, 1–7 (2000).

[23] Dorogovtsev, S. N., Mendes, J. F. F., and Samukhin, A. N., *Phys. Rev. Lett.* **85**, 4633–4636 (2000).

A Figures

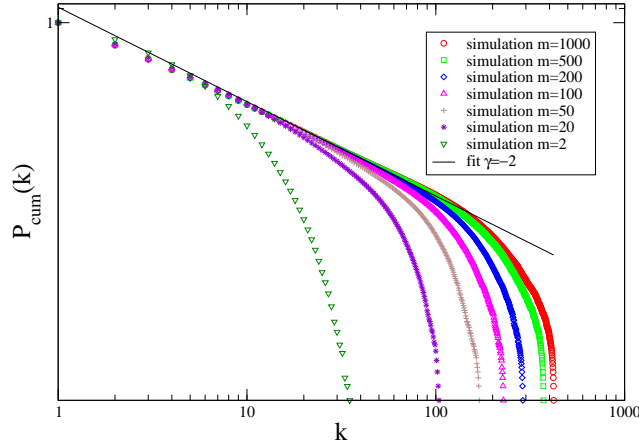


Fig. A.1. The node degree cumulative probability distribution obtained in the simulations of the network growth is displayed for the values of the filtering subset $m = 2, 20, 50, 100, 200, 500$ and 1000 . The cumulative probability distributions follow the power law for a range of node degrees, whereas for large values they exhibit a quickly decaying tail. The straight line represents the fit of the power law behavior with the exponent -2 .

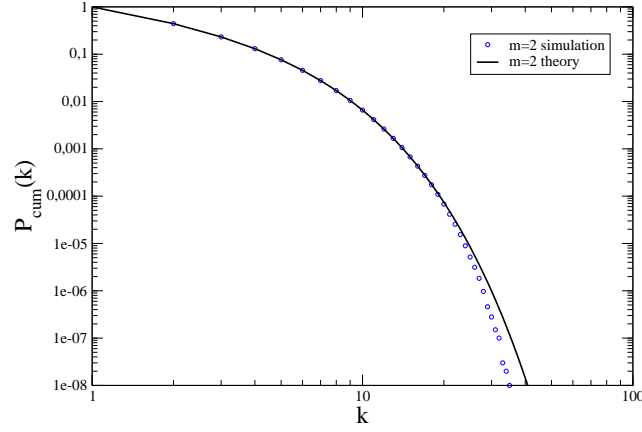


Fig. A.2. The comparison of the node degree cumulative probability distributions obtained by simulation (circles) and theoretical calculations (full line) for $m = 2$. Theoretical and simulational distributions are in excellent agreement although the power law aspect of the distribution is very weakly expressed. The disagreement in the tail of the distributions may be attributed to the finite size effect in the simulational data.

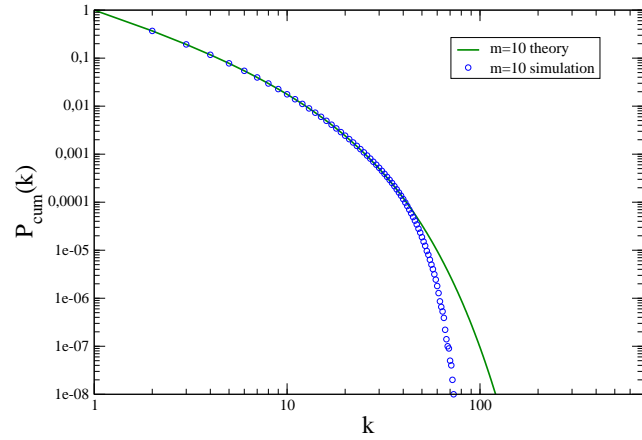


Fig. A.3. The comparison of the node degree cumulative probability distributions obtained by simulation (circles) and theoretical calculations (full line) for $m = 10$. Excellent agreement of the two distributions is evident. The disagreement at the largest node degrees is attributed to the final size effects in the simulational distribution.

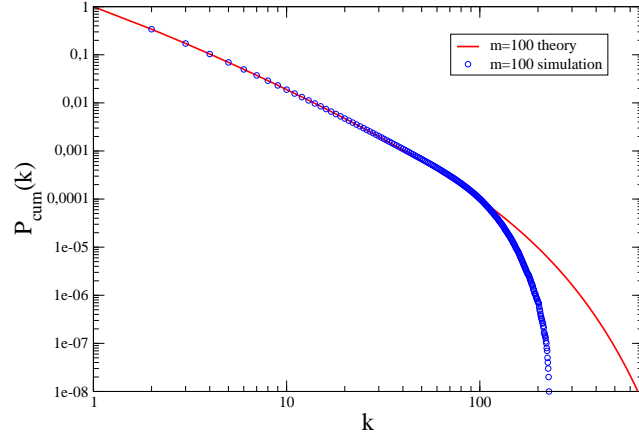


Fig. A.4. The comparison of the node degree cumulative probability distributions obtained by simulation (circles) and theoretical calculations (full line) for $m = 100$. The two distributions are in excellent agreement, except for the largest values of node degrees. The disagreement in this limit is attributed to the final size of the simulational data set.

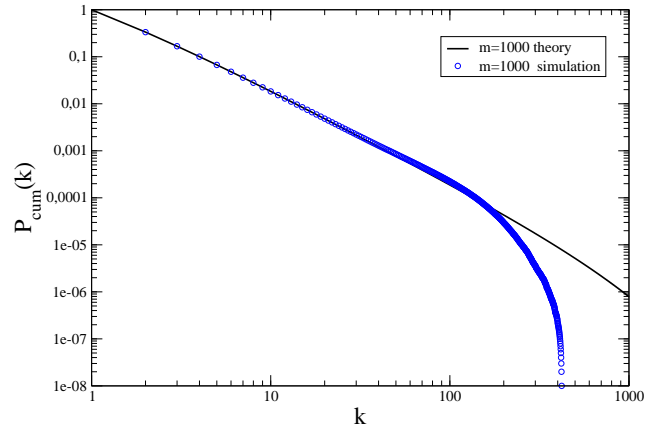


Fig. A.5. The comparison of the node degree cumulative probability distributions obtained by simulation (circles) and theoretical calculations (full line) for $m = 1000$. Excellent agreement of the two distributions is absent only for the largest values of the node degrees which is explained as a consequence of a finite data set obtained by simulation.

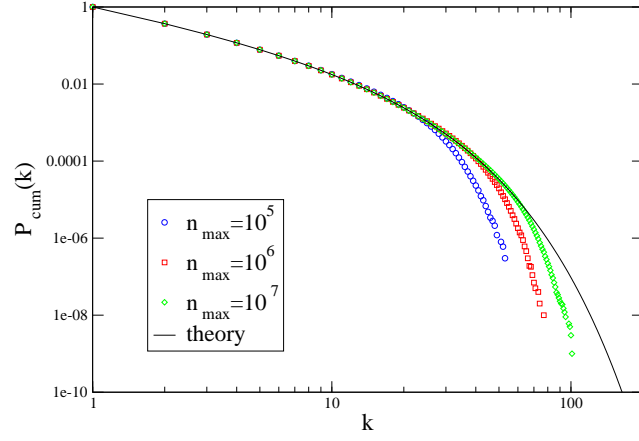


Fig. A.6. The asymptotical approach of the simulated cumulative degree probability distribution for sample size $m = 10$, and different maximum network sizes $n_{max} \in \{10^5, 10^6, 10^7\}$ to the theoretically obtained one. Other sample sizes m also exhibit same behavior.

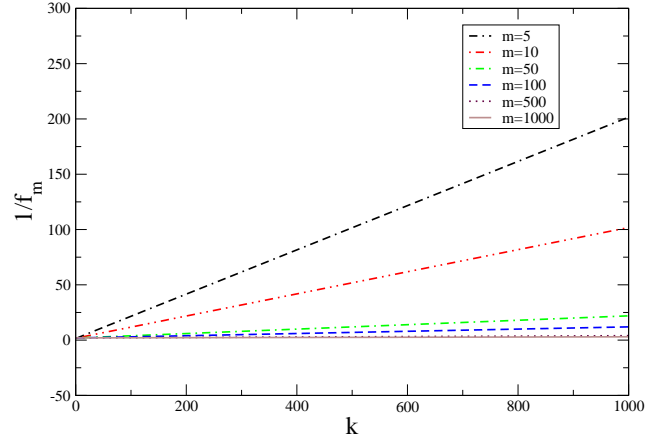


Fig. A.7. The inverse of the function f_m as the function of the node degree for the filtering subset sizes $m = 5, 10, 50, 100, 500$, and 1000 . The graphs clearly show the linearity of functions $1/f_m$.

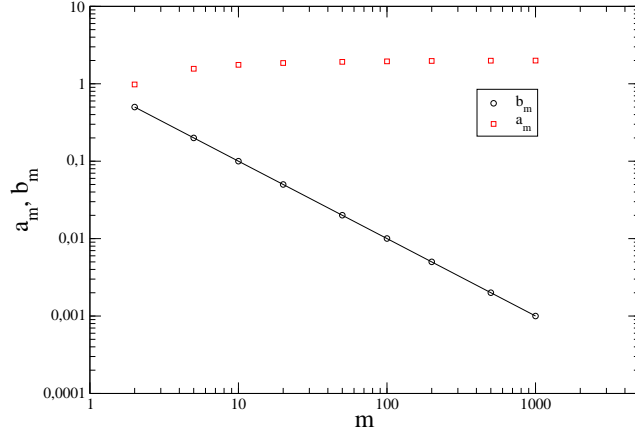


Fig. A.8. The values of the coefficients a_m and b_m for 9 different values of m . The coefficient a_m tends to the value 2 as m increases. The coefficient b_m fits very well to the function $1/m$ as seen from the best fit function.

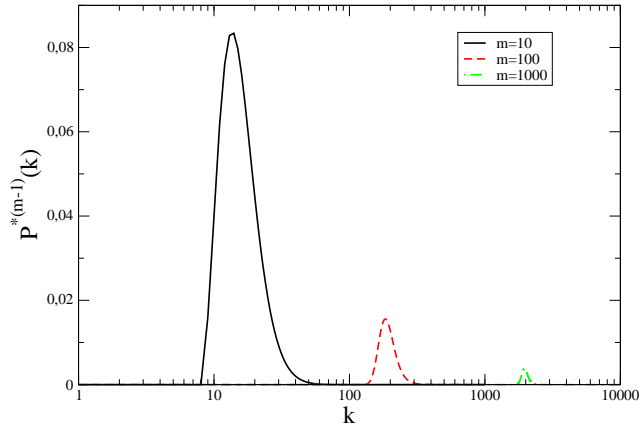


Fig. A.9. Probability distribution $P^{*(m-1)}(k)$ displayed for three values of $m = 10, 100$, and 1000. The localized character of the distributions is clearly visible from the graphs.